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Noncommuting contractions of oscillators with constant force

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Abstract

We find two noncommuting contractions of the Lie algebras $\mathsf{u}(N)$ and $\mathsf{gl}(N, \Re)$, realized as the symmetry algebras of N-dimensional isotropic harmonic and repulsive oscillators of spring constant $k \in \Re$, with a constant force of magnitude f. The contraction limit to the symmetry algebra of the N-dimensional free system is $(k, f) \to (0, 0)$. We take two paths in this plane, determined by the order of contraction of the two parameters, and show that they yield two closely related—but distinct—Euclidean-type symmetry algebras for the common contracted system. We also show briefly how the wavefunctions of the one-dimensional harmonic oscillator reduce to plane waves along the above two paths.

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1. Introduction

The transition from one system to another is well known in many branches of physics, such as the $c \to \infty$ limit of relativity to Newtonian mechanics, or the $\hbar \to 0$ limit of quantum to classical mechanics. In the theory of Lie algebras and groups, the Inönü–Wigner contraction [1] is paradigmatic in that it shows how the structure of the corresponding symmetry algebras and its representations changes when one parameter approaches a given limit. There is a large literature on group and algebra contractions with many applications [2].

In this paper, we examine a limit of Lie algebras under contractions with *two* parameters, where we can see that distinct paths in the parameter plane to a given limit can lead to distinct contracted algebras. The algebras we consider are the symmetry algebras of oscillators (either harmonic or repulsive) with a constant external force. The two parameters are the spring

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constant k and the magnitude $f = |\mathbf{f}|$ of a constant force. In this (k, f) plane, we examine two paths leading to the symmetry algebra of the free particle at (0, 0).

In section 2, we recall the symmetry algebras of the N-dimensional harmonic and repulsive oscillators, and their joint standard contraction $k \to 0$ to a symmetry algebra of the free particle. In section 3, we add the constant force to the oscillators and show that their symmetry algebra is preserved. In section 4, we subject the symmetry algebras of the oscillators to the contraction $k \to 0$, which result in the symmetry algebra of the free-fall system with the constant force. Finally, in section 5 we complete this nonstandard path through letting $f \to 0$, and thus find a *second* Lie algebra which is also the symmetry of the free particle. In section 6 we quote and discuss some results on how the discrete and normalized wavefunctions of the harmonic oscillator reduce to continuous plane waves along the above two contraction paths.

In section 7, we comment on the end result of the two contraction paths being two closely related—but distinct—symmetry Lie algebras for their common limit system. This phenomenon of noncommutativity of distinct *deformation and contraction* limits reminds us of the phenomenon of *hysteresis*, so we can characterize it as DC *hysteresis*.

2. The symmetry of oscillators

The Hamiltonian

$$H_0 := \frac{\mathbf{p}^2}{2m} + \frac{k\mathbf{x}^2}{2} = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{kx_i^2}{2} \right), \qquad k \in \Re, \tag{1}$$

describes the N-dimensional isotropic harmonic oscillator when the spring constant is positive ($k > 0$), a repulsive oscillator (sometimes called 'inverted oscillator') when it is negative ($k < 0$), and for $k = 0$ it describes a free particle.

Let $\mathbf{x} = \{x_i\}_{i=1}^N$ and $\mathbf{p} = \{p_i\}_{i=1}^N$ denote the operators of position and momentum respectively, whose commutation relations, we recall, are

$$[x_j, p_k] = \mathrm{i}\delta_{j,k}(\lambda/2\pi)\hat{1}, \qquad [\hat{1}, x_j] = 0, \quad [\hat{1}, p_k] = 0, \tag{2}$$

where $\hat{1}$ is the identity operator and $\lambda/2\pi$ is the reduced wavelength in paraxial wave optics or, in quantum mechanics, the reduced Heisenberg constant $\hbar = h/2\pi$. This value distinguishes the unitary irreducible representations of the Heisenberg–Weyl Lie algebra w_N, which is spanned by the $2N + 1$ operators x_j, p_k and $\hat{1}$. We consider here *natural* units, where $\hbar = 1$.

The generators of the manifest symmetry of the Hamiltonian (1) are the self-adjoint operators [4]

$$J_{i,j} := x_i p_j - x_j p_i = -J_{j,i} = J_{i,j}^\dagger, \qquad 1 \leqslant i < j \leqslant N, \tag{3}$$

which integrate to joint rotations in the position and in the momentum spaces. Their commutators close with the structure of the orthogonal Lie algebra $\mathsf{so}(N)$:

$$[J_{i,j}, J_{k,l}] = \mathrm{i}(\delta_{i,k} J_{j,l} - \delta_{i,l} J_{j,k} - \delta_{j,k} J_{i,l} + \delta_{j,l} J_{i,k}), \tag{4}$$

and form a skew-symmetric tensor. On the other hand, the dynamical (or 'hidden') symmetries of (1) are generated by the self-adjoint operators

$$H_{i,j} := \frac{p_i p_j}{2m} + \frac{k}{2} x_i x_j = H_{j,i} = H_{i,j}^\dagger \qquad 1 \leqslant i \leqslant j \leqslant N. \tag{5}$$

These generate linear transformations between one position and one momentum coordinate, and transform under the previous $\mathsf{so}(N)$ as a rank-two symmetric tensor,

$$[J_{i,j}, H_{k,l}] = \mathrm{i}(\delta_{i,k} H_{j,l} + \delta_{i,l} H_{j,k} - \delta_{j,k} H_{i,l} - \delta_{j,l} H_{i,k}). \tag{6}$$

Under commutation, they close into the previous rotations (3),

$$[H_{i,j}, H_{k,l}] = \mathrm{i}\frac{k}{4m}(\delta_{i,k}J_{j,l} + \delta_{i,l}J_{j,k} + \delta_{j,k}J_{i,l} + \delta_{j,l}J_{i,k}). \tag{7}$$

In the harmonic case $k > 0$ the span of the $\frac{1}{2}N(N-1)$ operators (3) and the $\frac{1}{2}N(N+1)$ operators (5) is the unitary Lie algebra $\mathsf{u}(N)$. This algebra was identified long ago as the symmetry algebra of the N-dimensional isotropic harmonic oscillator [3, 4]. In the repulsive case $k < 0$, the generated Lie algebra is the noncompact $\mathsf{gl}(N, \Re)$, the real general linear algebra studied recently by Daboul [5]. In both cases, the oscillator Hamiltonian (1) is the trace $H_0 := \sum_{j=1}^{N} H_{j,j}$, which is the centre of each of the algebras. They thus decompose as the direct sums $\mathsf{u}(N) = \mathsf{u}(1) \oplus \mathsf{su}(N)$ and $\mathsf{gl}(N, \Re) = \mathsf{gl}(1, \Re) \oplus \mathsf{sl}(N, \Re)$. If we were to get rid of the factor $k/(4m)$ in (7) we would rescale the generators $H_{i,j} \mapsto \sqrt{4m/|k|}H_{i,j}$; we refrain to do so because we intend to use $\varepsilon := \sqrt{|k|/4m}$ as our first contraction parameter.

When the spring constant vanishes, $k \to 0^{\pm}$, both $\mathsf{u}(N)$ and $\mathsf{gl}(N, \Re)$ contract to a distinct algebra, which consists of the same $\mathsf{so}(N)$ subalgebra of J's, and a contracted set of new 'free Hamiltonian' generators (see [5, 6]) which are

$$H^{\mathrm{F}}_{i,j} := \lim_{k\to 0} H_{i,j} = \frac{p_i p_j}{2m}, \quad 1 \leqslant i \leqslant j \leqslant N, \qquad H^{\mathrm{F}}_0 := \sum_{j=1}^{N} H^{\mathrm{F}}_{j,j} = \frac{\mathbf{p}^2}{2m}. \tag{8}$$

These continue to transform as a rank-two symmetric tensor (6), but now commute among themselves because for $k = 0$ the right-hand side of (7) is zero. The contractions of the oscillator algebras to the free case are thus

$$\lim_{k\to 0^+} \mathsf{u}(N) = \mathsf{i}^2_N\mathsf{so}(N) = \lim_{k\to 0^-} \mathsf{gl}(N, \Re), \tag{9}$$

where we denote the rank-two *inhomogeneous* orthogonal algebra by

$$\mathsf{i}^2_N\mathsf{so}(N) := \mathsf{i}^2_N \ni \mathsf{so}(N), \quad \text{with} \quad \mathsf{i}^2_N := \{p_i p_j | i, j = 1, \ldots, N\}, \tag{10}$$

which is the semidirect sum of the orthogonal subalgebra $\mathsf{so}(N)$ with the $\frac{1}{2}N(N+1)$-dimensional Abelian ideal i^2_N of the operators (8) that are quadratic in momentum. The latter generate *slants* of phase space, i.e., $(\mathbf{x}, \mathbf{p}) \mapsto (\mathbf{x} + \alpha\mathbf{V}\mathbf{p}, \mathbf{p})$, with a symmetric matrix $\mathbf{V}$ and the evolution parameter α. When $\mathbf{V} = \mathbf{1}$, this is the result of inertial free flight in mechanics, or free-light propagation in a homogeneous medium in paraxial optics. The symmetry algebra of the free system still has N^2 generators. As emphasized in [5], *all* $H_{i,j}$'s are realizations of the symmetric generators of $\mathsf{u}(N)$, multiplied implicitly by the contraction parameter $\varepsilon = \sqrt{|k|/4m}$. This is why their commutators in (7) vanish in the limit $k \to 0$, and consequently the above contraction is of the standard Inönü–Wigner type [1, 4].

3. Addition of a constant force

We now *deform* the isotropic oscillator by adding a constant force $\mathbf{f}$ of magnitude $f = |\mathbf{f}|$ (this is a true physical, not fictitious force). We are free to choose the Nth coordinate x_N of the system along this force, so that the original classical Hamiltonian (1) becomes

$$H^{(k,f)} := H_0 - f x_N = \frac{\mathbf{p}^2}{2m} + \frac{k\mathbf{x}^2}{2} - f x_N. \tag{11}$$

Because of the constant force, space has now a preferred direction, and the full rotational symmetry of the original system is broken. However, for $k \neq 0$ we can recover this symmetry simply by shifting the origin of the position coordinate system, $x_i \mapsto \xi_i := x_i - \delta_{i,N} f/k$, in the Hamiltonian (2). This corresponds to the following linear combination with the central element of the Heisenberg–Weyl algebra

$$x_i \mapsto \xi_i = x_i - \delta_{i,N}\frac{f}{k}\hat{1}. \qquad (12)$$

In terms of the new position operators ξ_i, the deformed Hamiltonian operator (11) now looks exactly as the original Hamiltonian (1), except for a constant additive term $f^2/2k$,

$$H^{(k,f)} := \frac{\mathbf{p}^2}{2m} + \frac{k\xi^2}{2} - \frac{f^2}{2k}\hat{1}. \qquad (13)$$

The generators of the new symmetry algebra are then linear combinations of the old generators $J_{i,j}$ and $H_{i,j}$, plus the w_N generators x_i, p_i and $\hat{1}$, in (2). Algebraically, this places the symmetry Lie algebras of the previous section, $\mathsf{u}(N)$, $\mathsf{gl}(N,\Re)$ or $\mathsf{i}^2_N\mathsf{so}(N)$, within larger *non*-invariance algebras, $\mathsf{w}_N\mathsf{u}(N)$, $\mathsf{w}_N\mathsf{gl}(N,\Re)$ or $\mathsf{w}_N\mathsf{i}^2_N\mathsf{so}(N)$ respectively. The symmetry algebras of the shifted oscillators are *subalgebras* of the w_N-extended symmetry algebras of the previous section, with the same structure as before. The new generators have the same expressions in terms of the shifted operators ξ_i as the previous ones had in terms of x_i, namely

$$J^{(k,f)}_{i,j} := \xi_i p_j - \xi_j p_i = J_{i,j}, \quad 1 \leqslant i < j < N, \quad \text{as in (3)}, \qquad (14)$$

$$J^{(k,f)}_{i,N} := \xi_i p_N - \xi_N p_i = J_{i,N} + \frac{f}{k}p_i, \quad 1 \leqslant i < N, \qquad (15)$$

$$H^{(k,f)}_{i,j} := \frac{p_i p_j}{2m} + \frac{k}{2}\xi_i\xi_j = H_{i,j}, \quad 1 \leqslant i \leqslant j < N, \quad \text{as in (5)}; \qquad (16)$$

$$H^{(k,f)}_{i,N} := \frac{p_i p_N}{2m} + \frac{k}{2}\xi_i\xi_N = H_{i,N} - \frac{1}{2}f x_i, \quad 1 \leqslant i < N, \qquad (17)$$

$$H^{(k,f)}_{N,N} := \frac{p_N p_N}{2m} + \frac{k}{2}\xi_N\xi_N = H_{N,N} - f x_N + \frac{f^2}{2k}\hat{1}. \qquad (18)$$

4. The symmetry of the free-fall system

When the constant force vanishes, $f \to 0$, the symmetry algebras of the oscillators with constant force contract uneventfully to those of section 1, as is clear from (14)–(18); and when $k \to 0$, the contraction follows as in equation (9) via the standard Inönü–Wigner procedure. Our purpose in this paper is to consider the alternative path illustrated in figure 1, namely to contract first the oscillators through $k \to 0$, to a system subject only to the constant force f, i.e., a *free-fall* system, and *then* let $f \to 0$, to regain the free particle. This second path will be examined now.

As long as $k \neq 0$, for large enough distances the oscillator force $-k\,\mathbf{x}$ eventually becomes stronger than any constant force $\mathbf{f}$, no matter how small k is. But as we approach $k = 0$, the shift f/k in the Nth position coordinate ξ_N in (12) becomes infinite, even though the Hamiltonian $H^{(k,f)}$ in (13), which is equivalent to $H^{(k,f)}$ in (11), remains finite. The constant term $f^2/2k$ in (13) simply yields a fictitious singularity; on the other hand, the new generators in (15) and (18) would blow up in the limit $k \to 0$. To make these generators finite, we contract them through multiplication by the factor k.

In contrast to the standard Inönu–Wigner contraction [1] of section 2, here the contraction procedure is *nonstandard*, since we are essentially multiplying different generators of $\mathsf{u}(N)$ and $\mathsf{gl}(N,\Re)$ by three different powers of the contraction parameter ε. As we mentioned earlier, $H_{i,j}$ already include an implicit multiplication by a linear power of ε. Hence $kH^{(k,f)}_{N,N}$ is essentially multiplied by ε^3, whereas $kJ^{(k,f)}_{i,N}$ are multiplied by ε^2.

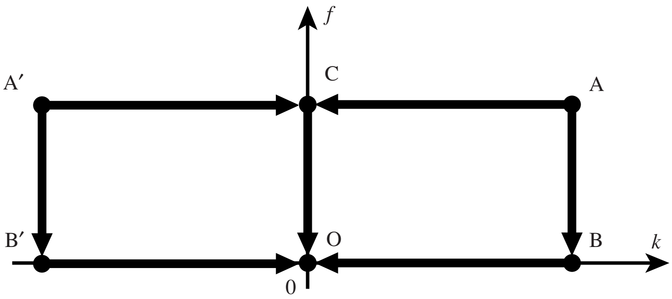

Figure 1. The (k, f) plane of contraction parameters for the Lie algebras that are symmetries of N-dimensional oscillators. The left half-plane $k < 0$ corresponds to the orbit of repulsive oscillators with a symmetry algebra $\mathsf{gl}(N, \Re)$, while the right half-plane $k > 0$ corresponds to the orbit of harmonic oscillators with a symmetry $\mathsf{u}(N)$. The f-axis contains the limit orbit $k = 0$ $(f \neq 0)$ of the symmetry algebra of free-fall systems, and the free system at $(k, f) = (0, 0)$. We follow two distinct contraction paths to arrive at the symmetry algebra of the free system, illustrating the '*DC hysteresis*' phenomenon in the parameter plane (k, f). The standard Inönü–Wigner contractions follow the paths $A \to B \to O$ and $A' \to B' \to O$ to the symmetry of rotations and cross-slants of position space and forming an $\mathsf{i}^2_N\mathsf{so}(N)$ algebra. On the other hand, the nonstandard contraction along the paths $A \to C \to O$ and $A' \to C \to O$ first contracts $k \to 0$ to the broken symmetry of the constant-force (free-fall) system at C, which includes translations, and ends up with a distinct symmetry algebra for the free system at O.

We thus write the generators of the free-fall system in terms of the generators of the free system (8), indicating the contracted generators by a tilde (cf (14)–(18)),

$$J^{(0,f)}_{i,j} = J_{i,j}, \quad 1 \leqslant i < j < N, \quad \text{as in (14)}, \tag{19}$$

$$\widetilde{J}^{(0,f)}_{i,N} := \lim_{k\to 0} k J^{(k,f)}_{i,N} = f p_i, \quad 1 \leqslant i < N, \tag{20}$$

$$H^{(0,f)}_{i,j} = H^{\mathrm{F}}_{i,j}, \quad 1 \leqslant i \leqslant j < N, \quad \text{as in (16)}, \tag{21}$$

$$H^{(0,f)}_{i,N} = H^{\mathrm{F}}_{i,N} - \tfrac{1}{2} f x_i, \quad 1 \leqslant i < N, \tag{22}$$

$$\widetilde{H}^{(0,f)}_{N,N} := \lim_{k\to 0} k H^{(k,f)}_{N,N} = \tfrac{1}{2} f^2. \tag{23}$$

To evince the structure of the contracted symmetry algebra (19)–(23) for (k=0, f≠0), we first note that all $(N-1)^2$ generators in the first $(N-1)$ coordinates $\{x_i\}_{i=1}^{N-1}$ contract as before (cf equation (9)), to the rank-two inhomogeneous orthogonal algebra in $(N-1)$ dimensions, $\mathsf{i}^2_{N-1}\mathsf{so}(N-1)$, while the $2(N-1)+1$ operators,

$$\widetilde{J}^{(0,f)}_{i,N}, \quad H^{(0,f)}_{i,N}, \quad 1 \leqslant i < N, \qquad \widetilde{H}^{(0,f)}_{N,N} = \tfrac{1}{2} f^2, \tag{24}$$

now form a Heisenberg–Weyl subalgebra w_{N-1}. The commutation relations to check are

$$\big[\widetilde{J}^{(0,f)}_{i,N}, \widetilde{J}^{(0,f)}_{j,N}\big] = 0, \quad \big[H^{(0,f)}_{i,N}, H^{(0,f)}_{j,N}\big] = 0, \quad \big[\widetilde{J}^{(0,f)}_{j,N}, H^{(0,f)}_{k,N}\big] = \mathrm{i}\delta_{j,k}\widetilde{H}^{(0,f)}_{N,N}, \tag{25}$$

cf (2). These w_{N-1} generators also transform covariantly under the previous $\mathsf{so}(N-1)$ algebra; however, they do *not all* commute with the previous rank-two inhomogeneous Abelian subalgebra i^2_{N-1} of generators $H^{(0,f)}_{i,j}$, $1 \leqslant i \leqslant j < N$. Instead,

$$\begin{aligned}&\big[H^{(0,f)}_{j,k}, \widetilde{J}^{(0,f)}_{l,N}\big] = 0,\\ &\big[H^{(0,f)}_{j,k}, \widetilde{H}^{(0,f)}_{N,N}\big] = 0,\end{aligned} \qquad \big[H^{(0,f)}_{j,k}, H^{(0,f)}_{l,N}\big] = \frac{\mathrm{i}}{4m}\big(\delta_{k,l}\widetilde{J}^{(0,f)}_{j,N} + \delta_{j,l}\widetilde{J}^{(0,f)}_{k,N}\big). \tag{26}$$

The two subalgebras, w_{N-1} and i^2_{N-1}, mesh into an algebra which is *not* their direct sum, but a *semidirect* sum

$$\mathsf{w}_{N-1}\mathsf{i}^2_{N-1} := \mathsf{w}_{N-1} \ni \mathsf{i}^2_{N-1}, \tag{27}$$

where w_{N-1} transforms under the Abelian group generated by i^2_{N-1}. The action can be understood quite naturally when we identify them as

$$\begin{aligned} &\text{'position'} \quad X_i \sim H^{(0,f)}_{i,N} = p_i p_N/2m - \tfrac{1}{2} f x_i, \\ &\text{'momentum'} \quad P_i \sim \widetilde{J}^{(0,f)}_{i,N} = f p_i, \qquad 1 \leqslant i < N, \end{aligned} \tag{28}$$

and recall that the commuting $H^{(0,f)}_{i,j}$'s generate the symmetric slants in the i–j planes, $X_l \mapsto X_l + \alpha(\delta_{i,l} P_j + \delta_{j,l} P_i)$, $P_l \mapsto P_l$, that we commented upon for the free particle at the end of section 2.

The structure of the symmetry algebra of the free-fall systems (19)–(23) obtained by the contraction of both the harmonic and repulsive oscillator algebras with constant force, equations (14)–(18), is therefore given by the following direct sum:

$$\mathsf{g}^{(0,f)} = \left(\mathsf{w}_{N-1}\mathsf{i}^2_{N-1}\right)\mathsf{so}(N-1) := \left(\mathsf{w}_{N-1}\mathsf{i}^2_{N-1}\right) \ni \mathsf{so}(N-1). \tag{29}$$

The full symmetry algebra (29) still has N^2 generators, but its structure is distinct from that of the precontracted algebras $\mathsf{u}(N)$ or $\mathsf{gl}(N, \Re)$.

Comparing the symmetry algebra of the free-fall system (29) with that of the free case, $\mathsf{i}^2_N\mathsf{so}(N)$ in (9), we see that the latter is broken along the x_N direction. There remains invariant the $\mathsf{i}^2_{N-1}\mathsf{so}(N-1)$ subalgebra that commutes with the coordinate x_N, and has $(N-1)^2$ generators. The other $2N-1$ generators that act on x_N were contracted to the Heisenberg–Weyl algebra w_{N-1}.

5. 'Deformation–contraction' (DC) hysteresis

We now close the second path of figure 1 by taking the limit $f \to 0$ to the free particle. Straightforwardly, in this limit the generators (19)–(23) become

$$J^{(0,0)}_{i,j} = J_{i,j}, \qquad 1 \leqslant i < j < N, \qquad \text{as in (19)}, \tag{30}$$

$$H^{(0,0)}_{i,j} = H^{\mathrm{F}}_{i,j}, \qquad 1 \leqslant i < j \leqslant N, \qquad \text{as in (21) and (22)} \tag{31}$$

while the N generators $\widetilde{J}^{0,0}_{i,N}$ and $\widetilde{H}^{0,0}_{N,N}$ *vanish*. This leaves the N^2-N generators (30) and (31) in a Lie algebra

$$\left(\mathsf{i}_{N-1} \oplus \mathsf{i}^2_{N-1}\right)\mathsf{so}(N-1), \tag{32}$$

where the generators of i_{N-1} are $H^{(0,0)}_{i,N}$'s $(1 \leqslant i < N)$ that transform under $\mathsf{so}(N-1)$ as an $(N-1)$-dimensional vector. The rotational symmetry $\mathsf{so}(N)$ of the free system cannot be recovered when the contraction path passes through constant-force systems.

In the limit $f \to 0$, $\widetilde{J}^{(0,f)}_{i,N} = f p_i$ and $\widetilde{H}^{(0,f)}_{N,N} = f^2/2$ would vanish, and the dimension of the resulting algebra would be diminished by N. However, by multiplying these two generators by the factor $1/f$, we can save one of them. The constant-force symmetry generators, which span the w_{N-1} subalgebra, become

$$\widetilde{\widetilde{J}}^{(0,0)}_{i,N} := \lim_{f\to 0} \frac{1}{f} \widetilde{J}^{(0,f)}_{i,N} = p_i, \qquad 1 \leqslant i < N, \tag{33}$$

$$H^{(0,0)}_{i,N} = H^{\mathrm{F}}_{i,N} = \frac{p_i p_N}{2m}, \qquad 1 \leqslant i < N, \tag{34}$$

$$\overset{\approx}{H}{}^{(0,0)}_{N,N} := \lim_{f\to 0} \frac{1}{f} \widetilde{H}^{(0,f)}_{N,N} = 0. \tag{35}$$

Since $\overset{\approx}{J}{}^{(0,0)}_{i,N}$ now commutes with $H^{(0,0)}_{i,N}$, in the limit $f \to 0$, w_{N-1} contracts to an Abelian algebra, as in the classic $\hbar \to 0$ contraction. The symmetry algebra thus obtained has now $N-1$ generators (33) more, and one generator (35) less, $\hat{1}$ being absent.

The structure of this symmetry algebra is thus

$$\left(\mathsf{i}_{N-1} \oplus \mathsf{i}_{N-1} \oplus \mathsf{i}^2_{N-1}\right)\mathsf{so}(N-1), \tag{36}$$

where we see that the w_{N-1} subalgebra in (29) has decomposed into a direct sum of two i_{N-1} without the identity operator $\hat{1}$. The first i_{N-1} in (36) contains the generators of translations $\{p_i\}_{i=1}^{N-1}$, while the second i_{N-1} contains the generators of slants into the direction of x_N, namely $\{p_i p_N/2m\}_{i=1}^{N-1}$. In this way we obtain N^2-1 generators—one generator less than the N^2 generators of $\mathsf{i}^2_N\mathsf{so}(N)$, the symmetry algebra of the free system that we found using the standard Inönü–Wigner contraction path in (9). Not only are the structures of the algebras different, but so are the transformations they generate. There are no generators that will rotate x_N and p_N, but all other slant generators are present, and now the *bonafide* translations p_i. As could be anticipated, the full rotational symmetry $\mathsf{so}(N)$ of the free case has been lost and cannot be recovered.

6. Limits of wavefunctions and spectra

At the behest of one of the referees we annotate some results from [7], where we examine the one-dimensional harmonic oscillator with a constant force, following the energies and eigenfunctions of the Hamiltonian (11) under the standard and nonstandard contractions, to show the effects of the lost symmetry.

Recovering the physical value of $\hbar$, the discrete spectrum and eigenfunctions of the harmonic oscillator are

$$E^{(k,f)}_n = \left(n + \frac{1}{2}\right)\hbar\omega - f^2/2k, \quad n \in \{0, 1, 2, \ldots\} \tag{37}$$

$$\psi^{(k,f)}_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n! a}} \exp\left[-\frac{1}{2}(\xi/a)^2\right] H_n(\xi/a), \tag{38}$$

where $H_n(z)$ are Hermite polynomials,

$$\omega := \sqrt{k/m}, \qquad \xi := x - f/k, \quad a := \sqrt{\hbar/\sqrt{km}}, \tag{39}$$

and we assume that $f > 0$. The standard limit, first for $f \to 0^+$ and second for a sequence $\{n\}$ of oscillators with spring constants $k_n = m\omega_n^2 \to 0^+$, such that the energy is kept fixed at $E^{(k_n,0)} = \left(n + \frac{1}{2}\right)\hbar\omega_n = p^2/2m$, yields the well-known parity-classified, Dirac-normalized free eigenfunctions [8]

$$\phi^{\pm}_p(x) = \frac{1}{\sqrt{\pi}} \begin{cases} \cos(px/\hbar), & p \geqslant 0, \\ \sin(px/\hbar), & p > 0. \end{cases} \tag{40}$$

On the other hand, when we follow the nonstandard contraction, the same sequence $k_n \to 0$, $f > 0$, with the classical turning point $x^{\rm tur}_E = -E/f$ for energy $E \in \Re$, we find the limit to the Airy functions of free fall in the negative-x direction,

$$\lim_{n\to\infty} k_n^{-1/4} \psi^{(k_n,f)}_n\left(x - x^{\rm tur}_E\right) = 2m^{1/12}\hbar f^{-1/6} \mathrm{Ai}((2mf/\hbar^2)^{1/3}(x + E/f)). \tag{41}$$

Finally, when $f \to 0^+$, negative-energy eigenfunctions vanish, $\phi_{E<0}(x) \underset{f\to 0}{\widetilde{\quad}} 0$, while the positive-energy states behave as

$$\phi_{E>0}(x) \underset{f\to 0}{\widetilde{\quad}} \frac{1}{\sqrt{2\pi}}((\cos\chi + \sin\chi)\cos(px/\hbar) + (\cos\chi - \sin\chi)\sin(px/\hbar)), \tag{42}$$

where $\chi := \sqrt{m}(2E)^{3/2}/(3\hbar f) \to \infty$ and $\phi_{E=0}(x) \underset{f\to 0}{\widetilde{\quad}} \pi^{-1/2}$. Hence, while (42) are solutions to the free Schrödinger equation, their parity is not recovered, but remains indeterminate. The detailed proofs of (41) and (42) are nontrivial [7] and lie outside the scope of this paper, which is the analysis of the non-commutativity in a class of two-parameter Lie algebra contractions.

7. Conclusions

DC hysteresis has been examined here as realized among well-known physical systems: harmonic and repulsive oscillators, free fall and the free particle in N dimensions. The contraction along the first path is a standard Inönü–Wigner contraction, while the contraction along the second path is nonstandard since, as explained in the text, it requires multiplication by three different powers of the contraction parameter $\varepsilon = \sqrt{|k|/4m}$, namely ε, ε^2 and ε^3, of the generators $H_{i,j}$ $(i < j \leqslant N)$, $J_{i,N}$ $(i < N)$ and $H_{N,N}$, respectively. On this model we have seen that symmetry lost on one leg of a multiparameter contraction path is never fully regained.

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